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A second order constitutive theory for hyperelastic materials

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Abstract

The second order constitutive equation for a hyperelastic material with arbitrary symmetry is derived. In developing a second order theory, it is necessary to be discriminating in the choice of measures of deformation. Here the derivation is done in terms of the Biot strain, which has a direct physical interpretation in that its eigenvalues are the principal extensions of the deformation. The constitutive equation is specialized for the cases of isotropy and transverse isotropy. The isotropic equation derived here is compared with equations obtained by other authors in terms of the displacement gradient and the Green strain.

1. Introduction

The purpose of this paper is to present a derivation of the second order constitutive equation for a hyperelastic material with arbitrary material symmetry. This general constitutive equation is then specialized for the material symmetries of isotropy and transverse isotropy.

Various authors have previously derived second order constitutive equations for isotropic elastic materials. These include theories in which the displacement gradient was used as the measure of deformation for hyperelastic materials (see, e.g., Rivlin, 1953; Murnaghan, 1937; Toupin and Bernstein, 1961; Haughton and Lindsay, 1993, 1994) and Cauchy elastic materials (see, e.g., Sheng, 1955). A number of these treatments are summarized in the *Nonlinear Field Theories of Mechanics* (Truesdell and Noll, 1965).

Constitutive equations that are second order in a strain measure, rather than the displacement gradient, have the advantage that the condition of material frame indifference can be easily satisfied. Such a second order constitutive equation was presented by Murnaghan (1951), who used the Green strain to derive a constitutive equation for isotropic hyperelastic material.

The derivation presented in this paper is done in terms of the Biot strain. Unlike the situations in classical infinitesimal elastic and in finite elasticity, the choice of strain measure is important in

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the derivation of a second order theory.[1] In the classical linear theory of elasticity, the choice of strain measure is inconsequential because, to first order in the displacement gradient, all measures of strain reduce to the symmetric part of the displacement gradient. In finite elasticity, the use of any strain measure that is an isotropic function of the Green deformation tensor is equivalent to the use of any other (Truesdell and Toupin, 1960). In contrast, a constitutive equation that is second order in one strain measure will not be second order in another, and the resulting expressions are models of different mechanical behaviors. The Biot strain is selected for use here because it has a clear physical interpretation: the eigenvalues of the Biot strain are the principal extensions.

A more detailed discussion of strain measures appears in Section 2, which also includes a brief review of derivatives of tensor valued functions of tensors and standard results on kinematics and stress constitutive equations. The derivation of the second-order constitutive equation for hyperelastic material with arbitrary symmetry is presented in Section 3. The resulting formulas for the Cauchy and Piola–Kirchhoff stresses are presented in terms of derivatives of the strain energy function with respect to the principal invariants appropriate to the symmetry of the material. These formulas are applied to the special cases of isotropy and transverse isotropy in Section 4.

Section 5 presents an examination of constitutive equations formulated in terms of two other measures of deformation: the displacement gradient and the Green strain. The constitutive equation for a material with arbitrary symmetry that is second order in the displacement gradient is derived; then it is specialized for isotropic materials and compared with constitutive equations previously derived by others. The constitutive equation which is second order in Green strain (obtained by Murnaghan) is considered, and its relation to the constitutive equation derived here in terms of Biot strain is established. The paper closes in Section 6 with a brief summary of the results.

2. Background

This section contains a review of the background information needed for the remainder of the paper. Standard terminology is used throughout (for example, see Gurtin, 1984; Ogden, 1984).

2.1. *Derivatives*

The second order stress constitutive equations will be derived by expanding the finite hyperelastic constitutive equation in the strain. Here the standard definitions and results associated with first and second derivatives of tensor valued functions are summarized (see, e.g., Gurtin, 1984; Dieudonne, 1960, for a rigorous presentation).

Let $\mathbf{M}$ be a function which maps tensors into tensors. (By tensor here I mean what is often termed a second order tensor. But, to avoid confusion, "second-order" will only be used in the context of second-order expansions and constitutive equations.) Then, if

[1] This has been clearly recognized in the past: for example, Ogden (1984, p. 350) pointed out that "in a second order theory it is important to distinguish between measures of deformation".

$$\lim_{\mathbf{X}\to\mathbf{0}} \frac{|\mathbf{M}(\mathbf{X})|}{|\mathbf{X}|} = 0,$$

$\mathbf{M}(\mathbf{X})$ is said to approach zero faster than $\mathbf{X}$, and one writes

$$\mathbf{M}(\mathbf{X}) = o(\mathbf{X}) \quad \text{as } \mathbf{X} \to \mathbf{0}.$$

A tensor valued function of tensors $\mathbf{G}$ is said to be differentiable at $\mathbf{A}$ if there exists a linear transformation $D\mathbf{G}(\mathbf{A})$, called the derivative of $\mathbf{G}$ at $\mathbf{A}$, such that

$$\mathbf{G}(\mathbf{A}+\mathbf{B}) - \mathbf{G}(\mathbf{A}) = D\mathbf{G}(\mathbf{A})[\mathbf{B}] + o(\mathbf{B}) \quad \text{as } \mathbf{B} \to \mathbf{0}. \tag{2.1}$$

$D\mathbf{G}(\mathbf{A})[\mathbf{B}]$ is the value of the derivative $D\mathbf{G}(\mathbf{A})$ on the increment $\mathbf{B}$.

If two tensor valued functions $\mathbf{G}$ and $\mathbf{K}$ are differentiable at $\mathbf{A}$, their product

$$\mathbf{P}(\mathbf{A}) = \mathbf{G}(\mathbf{A})\mathbf{K}(\mathbf{A})$$

is also differentiable at $\mathbf{A}$, and

$$D\mathbf{P}(\mathbf{A})[\mathbf{B}] = D\mathbf{G}(\mathbf{A})[\mathbf{B}]\mathbf{K}(\mathbf{A}) + \mathbf{G}(\mathbf{A})D\mathbf{K}(\mathbf{A})[\mathbf{B}]. \tag{2.2}$$

Let the tensor valued function $\mathbf{G}$ be differentiable at $\mathbf{A}$, and let the tensor valued function $\mathbf{J}$ be differentiable at $\mathbf{G}(\mathbf{A})$. Then the derivative of the composition

$$\mathbf{X}(\mathbf{A}) = \mathbf{J}(\mathbf{G}(\mathbf{A})),$$

is given by the chain rule:

$$D\mathbf{X}(\mathbf{A})[\mathbf{B}] = D\mathbf{J}(\mathbf{G}(\mathbf{A}))[D\mathbf{G}(\mathbf{A})[\mathbf{B}]]. \tag{2.3}$$

The second derivative of a tensor valued function of a tensor $\mathbf{G}$ is the derivative of $D\mathbf{G}(\mathbf{A})$. Thus, if the first derivative of $\mathbf{G}$ is differentiable,

$$D^2\mathbf{G}(\mathbf{A}) = D\{D\mathbf{G}(\mathbf{A})\}. \tag{2.4}$$

It is often convenient to view the second derivative as the bilinear map

$$D^2\mathbf{G}(\mathbf{A})[\mathbf{B},\mathbf{C}] = D\{D\mathbf{G}(\mathbf{A})[\mathbf{B}]\}[\mathbf{C}].$$

If the second derivative of $\mathbf{G}$ is continuous, then it is symmetric in the increments, i.e.,

$$D^2\mathbf{G}(\mathbf{A})[\mathbf{B},\mathbf{C}] = D^2\mathbf{G}(\mathbf{A})[\mathbf{C},\mathbf{B}]. \tag{2.5}$$

For a composite function, the second derivative can be expressed in terms of the second derivatives of the component functions. Let $\mathbf{X}(\mathbf{A}) = \mathbf{J}(\mathbf{G}(\mathbf{A}))$ as above, and suppose that $\mathbf{J}$ and $\mathbf{G}$ are both twice differentiable. Then $\mathbf{X}$ is twice differentiable, and

$$D^2\mathbf{X}(\mathbf{A})[\mathbf{B},\mathbf{C}] = D\mathbf{J}(\mathbf{G}(\mathbf{A}))[D^2\mathbf{G}(\mathbf{A})[\mathbf{B},\mathbf{C}]] + D^2\mathbf{J}(\mathbf{G}(\mathbf{A}))[D\mathbf{G}(\mathbf{A})[\mathbf{B}], D\mathbf{G}(\mathbf{A})[\mathbf{C}]]. \tag{2.6}$$

2.2. *Kinematics*

Let B represent the body in a fixed reference configuration in which it is unloaded and at rest. A deformation $\mathbf{f}$ is a smooth one-to-one mapping that carries point $\mathbf{p} \in B$ into point $\mathbf{x} = \mathbf{f}(\mathbf{p})$. The deformation gradient

$$\mathbf{F}(\mathbf{p}) = \nabla \mathbf{f}(\mathbf{p})$$

is assumed to satisfy $\det \mathbf{F}(\mathbf{p}) > 0$. Unless required for clarity, the dependence of fields on $\mathbf{p}$ will be left implicit in the remainder of the paper.

By the polar decomposition theorem the deformation gradient $\mathbf{F}$ can be uniquely represented as

$$\mathbf{F} = \mathbf{R}\mathbf{U}, \tag{2.7}$$

where the rotation $\mathbf{R}$ is proper orthogonal, and the right stretch tensor $\mathbf{U}$ is positive definite symmetric.

The Cayley–Hamilton theorem states that every tensor satisfies its own characteristic equation, so $\mathbf{U}$ meets

$$\mathbf{U}^3 - \mathrm{I}\mathbf{U}^2 + \mathrm{II}\mathbf{U} - \mathrm{III}\mathbf{1} = \mathbf{0}, \tag{2.8}$$

where $\mathbf{1}$ is the identity tensor, and the principal invariants of $\mathbf{U}$ are

$$\begin{aligned} \mathrm{I} &= \operatorname{tr} \mathbf{U}, \\ \mathrm{II} &= \tfrac{1}{2}[(\operatorname{tr} \mathbf{U})^2 - \operatorname{tr} \mathbf{U}^2], \\ \mathrm{III} &= \det \mathbf{U} = \tfrac{1}{6}[(\operatorname{tr} \mathbf{U})^3 - 3(\operatorname{tr} \mathbf{U})(\operatorname{tr} \mathbf{U}^2) + 2 \operatorname{tr} \mathbf{U}^3]. \end{aligned} \tag{2.9}$$

The right Cauchy–Green deformation tensor is the square of the right stretch tensor:

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{U}^2. \tag{2.10}$$

The eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$ of $\mathbf{U}$ are called the principal stretches and describe the ratio of the deformed length to the original length of a material filament in the principal directions at a point. The eigenvalues of $\mathbf{C}$ are the squares of the eigenvalues of $\mathbf{U}$.

The displacement $\mathbf{u}$ is related to the deformation by

$$\mathbf{u}(\mathbf{p}) = \mathbf{f}(\mathbf{p}) - \mathbf{p},$$

so the displacement gradient $\mathbf{H} = \nabla \mathbf{u}$ is related to the deformation gradient by

$$\mathbf{H} = \mathbf{F} - \mathbf{1}. \tag{2.11}$$

The symmetric part of $\mathbf{H}$,

$$\mathbf{E} = \operatorname{sym} \mathbf{H} = \tfrac{1}{2}(\mathbf{H} + \mathbf{H}^{\mathrm{T}}), \tag{2.12}$$

is termed the elongation tensor (Truesdell and Toupin, 1960). When the displacement gradient is small, $\mathbf{E}$ is the infinitesimal strain and the skew part of $\mathbf{H}$,

$$\mathbf{W} = \operatorname{skw} \mathbf{H} = \tfrac{1}{2}(\mathbf{H} - \mathbf{H}^{\mathrm{T}}), \tag{2.13}$$

is the infinitesimal rotation.

2.3. *Measures of strain*

Two strain measures will be used in this paper: the Biot strain $\mathbf{E}_1$, defined as

$$\mathbf{E}_1 = \mathbf{U} - \mathbf{1}; \tag{2.14}$$

and the Green strain $\mathbf{E}_2$, given by

$$\mathbf{E}_2 = \tfrac{1}{2}(\mathbf{U}^2 - \mathbf{1}). \tag{2.15}$$

These strains are related by

$$\mathbf{E}_2 = \mathbf{E}_1 + \tfrac{1}{2}(\mathbf{E}_1)^2. \tag{2.16}$$

The eigenvalues of $\mathbf{E}_1$ are the principal extensions $\{\delta_1, \delta_2, \delta_3\}$, defined through the principal stretches as

$$\delta_i = \lambda_i - 1,$$

where, of course, $\delta_i > 0$ indicates lengthening of a material filament in the ith principal direction, and $\delta_i < 0$ corresponds to shortening.

The Green strain has been used extensively in the finite elasticity literature, and many other strain measures have also been found useful in various contexts. It has long been recognized that any uniquely invertible isotropic tensor function of $\mathbf{C}$ can be used as a measure of strain (e.g., see Ogden, 1984 or the discussion in Section 32 of Truesdell and Toupin, 1960), and that, in the context of finite elasticity, such strain measures are equivalent in the sense that a constitutive equation that can be expressed in terms of one such strain measure can also be expressed in terms of any other.[2]

In contrast, the use of different strain measures is typically *not equivalent* in this sense for the derivation of second order constitutive equations. This is because each strain measure may contain a different power of the stretch. As a specific example, consider the Biot and Green strains defined above. Expansion of the stress constitutive equation in $\mathbf{E}_1$ with retention of terms that are linear and quadratic in $\mathbf{E}_1$ will generate an expression that contains only linear and quadratic terms in the extensions. Expansion of the same stress constitutive equation in $\mathbf{E}_2$, retaining terms that are linear and quadratic in $\mathbf{E}_2$ will given an equation which contains third and fourth order terms in the extensions, as well as the terms which are first and second order in the extensions. Thus, even though the forms of the equations are identical on a superficial level, the second order expansions in $\mathbf{E}_1$ and $\mathbf{E}_2$ give different approximations to the finite elastic response (or in an alternative view, describe physically different materials).

In the derivations presented in this paper, the Biot strain will be used. As noted in *The Classical Field Theories* (p. 269, Truesdell and Toupin, 1960), the use of $\mathbf{U}$ for the description of strain is "attractive in that its proper numbers are exactly the principal stretches". The associated strain measure $\mathbf{E}_1$ has a direct correspondence to the physical notion of extension, and therefore can best

[2] Of course, the use of one particular strain measure may be much more convenient or make it possible to write the constitutive equation in an especially compact form.

reflect the physically intuitive notion of the elongation being small that is the basis of the derivation of a second order constitutive equation.

Finally, note that for the derivation of linear elastic constitutive equations the distinction among the various strain measures does not have to be made because, to first order, they are identical; specifically,

$$\mathbf{E}_2 = \mathbf{E}_1 + o(\mathbf{E}_1),$$

and

$$\mathbf{E}_1 = \mathbf{E}_2 + o(\mathbf{E}_2),$$

(see e.g., Hoger, 1993a).

2.4. *Stress*

The body is assumed to respond to deformations from the reference configuration in an elastic manner, so the constitutive equation for the Cauchy stress $\mathbf{T}$ can be written in terms of a response function $\hat{\mathbf{T}}$ as

$$\mathbf{T}(\mathbf{p}) = \hat{\mathbf{T}}(\mathbf{F}(\mathbf{p}), \mathbf{p}). \tag{2.17}$$

In order that the elastic response be independent of observer, the response function must satisfy

$$\hat{\mathbf{T}}(\mathbf{QF}) = \mathbf{Q}\hat{\mathbf{T}}(\mathbf{F})\mathbf{Q}^{\mathrm{T}} \tag{2.18}$$

for every deformation gradient $\mathbf{F}$ and every proper orthogonal tensor $\mathbf{Q}$.

The first Piola–Kirchhoff stress $\mathbf{S}$ is defined in terms of the Cauchy stress through

$$\mathbf{S} = (\det \mathbf{F})\mathbf{T}\mathbf{F}^{-\mathrm{T}} = (\det \mathbf{F})\hat{\mathbf{T}}(\mathbf{F})\mathbf{F}^{-\mathrm{T}} =: \hat{\mathbf{S}}(\mathbf{F}). \tag{2.19}$$

In terms of $\hat{\mathbf{S}}$, observer independence requires that

$$\hat{\mathbf{S}}(\mathbf{QF}) = \mathbf{Q}\hat{\mathbf{S}}(\mathbf{F}) \tag{2.20}$$

for all deformation gradients $\mathbf{F}$ and proper orthogonal $\mathbf{Q}$.

For an elastic solid, the material symmetry group at point $\mathbf{p}$ is the set of all proper orthogonal tensors $\mathbf{Q}$ that satisfy

$$\hat{\mathbf{T}}(\mathbf{F}(\mathbf{p})\mathbf{Q}, \mathbf{p}) = \hat{\mathbf{T}}(\mathbf{F}(\mathbf{p}), \mathbf{p}) \tag{2.21}$$

for all deformation gradients $\mathbf{F}$. The material symmetry group at $\mathbf{p}$ will be denoted by $\mathscr{G}_{\mathbf{p}}$. When combined with (2.21), observer independence implies that the response function $\hat{\mathbf{T}}$ is invariant under $\mathscr{G}_{\mathbf{p}}$; i.e., for all $\mathbf{Q} \in \mathscr{G}_{\mathbf{p}}$

$$\hat{\mathbf{T}}(\mathbf{QFQ}^{\mathrm{T}}) = \mathbf{Q}\hat{\mathbf{T}}(\mathbf{F})\mathbf{Q}^{\mathrm{T}}. \tag{2.22}$$

In terms of the response function for the Piola–Kirchhoff stress, (2.22) becomes

$$\hat{\mathbf{S}}(\mathbf{Q}\mathbf{F}\mathbf{Q}^{\mathrm{T}}) = \mathbf{Q}\hat{\mathbf{S}}(\mathbf{F})\mathbf{Q}^{\mathrm{T}}. \tag{2.23}$$

2.5. *The strain energy function*

A material is hyperelastic if there exists a scalar valued function $\bar{\sigma}$ such that

$$\bar{\mathbf{S}}(\mathbf{F}(\mathbf{p}), \mathbf{p}) = \frac{\partial \bar{\sigma}}{\partial \mathbf{F}}(\mathbf{F}(\mathbf{p}), \mathbf{p}), \tag{2.24}$$

where the gradient $\partial\bar{\sigma}/\partial\mathbf{F}$ is the unique tensor defined through

$$D\bar{\sigma}(\mathbf{F})[\mathbf{A}] = \frac{\partial \bar{\sigma}}{\partial \mathbf{F}}(\mathbf{F}) \cdot \mathbf{A} \tag{2.25}$$

(Gurtin, 1984). The function $\bar{\sigma}$ is called the strain energy or stored energy function. In some of the following calculations it will be convenient to write the gradient as

$$\nabla\bar{\sigma}(\mathbf{F}) := \frac{\partial \bar{\sigma}}{\partial \mathbf{F}}(\mathbf{F}).$$

In order that the principle of observer independence be satisfied, $\bar{\sigma}$ must meet

$$\bar{\sigma}(\mathbf{F}) = \bar{\sigma}(\mathbf{U}). \tag{2.26}$$

From this point on, $\hat{\sigma}$ will be used to denote the restriction of the strain energy function to the set of positive definite symmetric tensors, i.e.,

$$\hat{\sigma} := \bar{\sigma}|_{P\mathrm{sym}}.$$

It was shown by Hoger (1993b) that the Piola–Kirchhoff stress can be written in terms of $(\partial\hat{\sigma}/\partial\mathbf{U})(\mathbf{U})$ as

$$\mathbf{S} = \mathbf{R}\left[\frac{\partial \hat{\sigma}}{\partial \mathbf{U}}(\mathbf{U}) + \frac{1}{(\mathrm{I}\,\mathrm{II} - \mathrm{III})}\left\{\left(\mathbf{U}^2\frac{\partial \hat{\sigma}}{\partial \mathbf{U}}(\mathbf{U})\mathbf{U} - \mathbf{U}\frac{\partial \hat{\sigma}}{\partial \mathbf{U}}(\mathbf{U})\mathbf{U}^2\right)\right.\right.$$
$$\left.\left. - \mathrm{I}\left(\mathbf{U}^2\frac{\partial \hat{\sigma}}{\partial \mathbf{U}}(\mathbf{U}) - \frac{\partial \hat{\sigma}}{\partial \mathbf{U}}(\mathbf{U})\mathbf{U}^2\right) + \mathrm{I}^2\left(\mathbf{U}\frac{\partial \hat{\sigma}}{\partial \mathbf{U}}(\mathbf{U}) - \frac{\partial \hat{\sigma}}{\partial \mathbf{U}}(\mathbf{U})\mathbf{U}\right)\right\}\right], \tag{2.27}$$

where the invariants, defined in (2.9), are those of $\mathbf{U}$. By use of relation (2.19) the Cauchy stress is seen to be

$$\mathbf{T} = \frac{1}{\mathrm{III}(\mathrm{I}\,\mathrm{II} - \mathrm{III})}\mathbf{R}\left\{\mathbf{U}^2\frac{\partial \hat{\sigma}}{\partial \mathbf{U}}(\mathbf{U})\mathbf{U}^2 - \mathrm{I}\left(\mathbf{U}^2\frac{\partial \hat{\sigma}}{\partial \mathbf{U}}(\mathbf{U})\mathbf{U} + \mathbf{U}\frac{\partial \hat{\sigma}}{\partial \mathbf{U}}(\mathbf{U})\mathbf{U}^2\right)\right.$$
$$\left. + (\mathrm{I}^2 + \mathrm{II})\left(\mathbf{U}\frac{\partial \hat{\sigma}}{\partial \mathbf{U}}(\mathbf{U})\mathbf{U}\right) - \mathrm{III}\left(\mathbf{U}\frac{\partial \hat{\sigma}}{\partial \mathbf{U}}(\mathbf{U}) + \frac{\partial \hat{\sigma}}{\partial \mathbf{U}}(\mathbf{U})\mathbf{U}\right) + \mathrm{I}\,\mathrm{III}\frac{\partial \hat{\sigma}}{\partial \mathbf{U}}(\mathbf{U})\right\}\mathbf{R}^{\mathrm{T}}. \tag{2.28}$$

These expressions for $\mathbf{S}$ and $\mathbf{T}$ are completely general, and are valid for any objective strain energy function.

3. The second order constitutive equation for a hyperelastic material

In this section the form of the second order stress constitutive equation appropriate for hyperelastic materials with arbitrary symmetry is developed. The derivation will be carried out by expanding expression (2.28) for the Cauchy stress in the strain $\mathbf{E}_1 = \mathbf{U}-\mathbf{1}$. First, the appropriate expansion for $(\partial\hat{\sigma}/\partial\mathbf{U})(\mathbf{U})$ will be obtained; then that expansion, together with the required expressions for the scalar terms will be incorporated into (2.28) to produce the desired result.

3.1. *Expansion of the derivative of strain energy*

The strain energy $\hat{\sigma}$ is a scalar valued function of $\mathbf{U}$ which is invariant under the symmetry group of the material. So it can be expressed as a function of the set of basic polynomial invariants of the right stretch appropriate for the given symmetry since this set constitutes an integrity basis (Spencer, 1971; Adkins, 1960), and therefore a functional basis (Wineman and Pipkin, 1965; Spencer, 1971). This method yields the canonical form for the strain (Wineman and Pipkin, 1965; Pipkin and Wineman, 1962; Spencer, 1971), which will be expanded in a Taylors series to produce the constitutive equation for the stress.

Given the material symmetry of the material, let $\mathscr{I}_{\mathbf{U}} = \{\mathrm{I}_i(\mathbf{U})\}$ denote the complete set of the n basic polynomial invariance of $\mathbf{U}$ appropriate for that material symmetry. The strain energy is a scalar valued function of these invariants, so $\hat{\sigma}(\mathbf{U})$ is given by a smooth function σ of the invariants as

$$\hat{\sigma}(\mathbf{U}) = \sigma(\mathrm{I}_1(\mathbf{U}), \mathrm{I}_2(\mathbf{U}), \ldots, \mathrm{I}_n(\mathbf{U})) =: \sigma(\mathscr{I}_{\mathbf{U}}). \tag{3.1}$$

By the chain rule

$$\nabla\hat{\sigma}(\mathbf{U}) := \frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{U}) = \frac{\partial\sigma}{\partial\mathrm{I}_j}(\mathscr{I}_{\mathbf{U}})\frac{\partial\mathrm{I}_j}{\partial\mathbf{U}}(\mathbf{U}). \tag{3.2}$$

Note that, given the symmetry group of the material $(\partial\mathrm{I}_j/\partial\mathbf{U})(\mathbf{U})$ is easy to calculate. (Isotropy and transverse isotropy will be treated in Section 4.)

With $\mathbf{U} = \mathbf{1}+\mathbf{E}_1$, the gradient of the strain energy can be expanded as

$$\begin{aligned}\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{U}) &= \frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1}+\mathbf{E}_1)\\ &= \frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1}) + D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[\mathbf{E}_1] + \frac{1}{2}D^2\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[\mathbf{E}_1,\mathbf{E}_1] + o(\mathbf{E}_1^2).\end{aligned} \tag{3.3}$$

In order that the stress vanish at $\mathbf{F} = \mathbf{1}$, it is necessary that

$$\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1}) = 0. \tag{3.4}$$

Now consider the second term in the expansion; (3.2) can be used to write

$$D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{U})[\mathbf{E}_1] = D\left\{\frac{\partial\sigma}{\partial\mathrm{I}_j}(\mathscr{I}_\mathbf{U})\frac{\partial\mathrm{I}_j}{\partial\mathbf{U}}(\mathbf{U})\right\}[\mathbf{E}_1],$$

and application of the product rule gives

$$D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{U})[\mathbf{E}_1] = \left\{D\frac{\partial\sigma}{\partial\mathrm{I}_j}(\mathscr{I}_\mathbf{U})[\mathbf{E}_1]\right\}\frac{\partial\mathrm{I}_j}{\partial\mathbf{U}}(\mathbf{U}) + \frac{\partial\sigma}{\partial\mathrm{I}_j}(\mathscr{I}_\mathbf{U})\left\{D\frac{\partial\mathrm{I}_j}{\partial\mathbf{U}}(\mathbf{U})[\mathbf{E}_1]\right\}. \tag{3.5}$$

The individual derivatives in this expression are

$$D\frac{\partial\sigma}{\partial\mathrm{I}_j}(\mathscr{I}_\mathbf{U})[\mathbf{E}_1] = \left\{\frac{\partial}{\partial\mathrm{I}_k}\left(\frac{\partial\sigma}{\partial\mathrm{I}_j}(\mathscr{I}_\mathbf{U})\right)\right\}\left\{\frac{\partial\mathrm{I}_k}{\partial\mathbf{U}}(\mathbf{U})\cdot\mathbf{E}_1\right\} \tag{3.6}$$

and

$$D\frac{\partial\mathrm{I}_j}{\partial\mathbf{U}}(\mathbf{U})[\mathbf{E}_1] = \frac{\partial^2\mathrm{I}_j}{\partial\mathbf{U}^2}(\mathbf{U})[\mathbf{E}_1],$$

where, for each $j = 1$ to n $(\partial^2\mathrm{I}_j/\partial\mathbf{U}^2)(\mathbf{U})$ is a fourth order tensor with components defined through

$$\frac{\partial^2\mathrm{I}_j}{\partial\mathbf{U}^2}(\mathbf{U})[\mathbf{E}_1] = \frac{\partial^2\mathrm{I}_j}{\partial U_{kl}\,\partial U_{pm}}(\mathbf{U})(\mathbf{E}_1)_{pm}\,\mathbf{e}_k\otimes\mathbf{e}_l.$$

Recall that $\mathscr{I}_\mathbf{U}$ is the set of basic polynomial invariants, so $\mathscr{I}_\mathbf{U} = \{\mathrm{I}_i(\mathbf{U})\}$ for $i = 1$ to n; at $\mathbf{U} = \mathbf{1}$, $\mathscr{I}_\mathbf{1} = \{\mathrm{I}_i(\mathbf{1})\}$. With this notation, incorporation of (3.6) into (3.5) and evaluation of the result at $\mathbf{U} = \mathbf{1}$ gives the second term in (3.3):

$$\Psi_1[\mathbf{E}_1] := D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[\mathbf{E}_1] = \frac{\partial^2\sigma}{\partial\mathrm{I}_k\,\partial\mathrm{I}_j}(\mathscr{I}_\mathbf{1})\frac{\partial\mathrm{I}_j}{\partial\mathbf{U}}(\mathbf{1})\left\{\frac{\partial\mathrm{I}_k}{\partial\mathbf{U}}(\mathbf{1})\cdot\mathbf{E}_1\right\} + \frac{\partial\sigma}{\partial\mathrm{I}_j}(\mathscr{I}_\mathbf{1})\frac{\partial^2\mathrm{I}_j}{\partial\mathbf{U}^2}(\mathbf{1})[\mathbf{E}_1]. \tag{3.7}$$

The name $\Psi_1[\mathbf{E}_1]$ is assigned to $D(\partial\hat{\sigma}/\partial\mathbf{U})(\mathbf{1})[\mathbf{E}_1]$ to reduce the bulkiness of subsequent equations.

To obtain the third term in expansion (3.3), recall that

$$D^2\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{U})[\mathbf{E}_1,\mathbf{E}_1] = D\left\{D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{U})[\mathbf{E}_1]\right\}[\mathbf{E}_1].$$

Incorporation of eqns (3.5) and (3.6) yields

$$D^2\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{U})[\mathbf{E}_1,\mathbf{E}_1] = D\left\{\frac{\partial^2\sigma}{\partial\mathrm{I}_k\,\partial\mathrm{I}_j}(\mathscr{I}_\mathbf{U})\frac{\partial\mathrm{I}_j}{\partial\mathbf{U}}(\mathbf{U})\left(\frac{\partial\mathrm{I}_k}{\partial\mathbf{U}}(\mathbf{U})\cdot\mathbf{E}_1\right) + \frac{\partial\sigma}{\partial\mathrm{I}_j}(\mathscr{I}_\mathbf{U})\frac{\partial^2\mathrm{I}_j}{\partial\mathbf{U}^2}(\mathbf{U})[\mathbf{E}_1]\right\}[\mathbf{E}_1]. \tag{3.8}$$

The product rule, together with (3.6) and

$$D\left\{\frac{\partial^2\sigma}{\partial\mathrm{I}_k\,\partial\mathrm{I}_j}(\mathscr{I}_\mathbf{U})\right\}[\mathbf{E}_1] = \frac{\partial^3\sigma}{\partial\mathrm{I}_p\,\partial\mathrm{I}_k\,\partial\mathrm{I}_j}(\mathscr{I}_\mathbf{U})\left\{\frac{\partial\mathrm{I}_p}{\partial\mathbf{U}}(\mathbf{U})\cdot\mathbf{E}_1\right\} \tag{3.9}$$

can be used to rewrite (3.8). The resulting expression is then evaluated at $\mathbf{U}=\mathbf{1}$ to obtain the third term in expansion (3.3):

$$\begin{aligned}\Psi_2[\mathbf{E}_1,\mathbf{E}_1] &:= D^2\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[\mathbf{E}_1,\mathbf{E}_1]\\ &=\frac{\partial^3\sigma}{\partial\mathrm{I}_p\,\partial\mathrm{I}_k\,\partial\mathrm{I}_j}(\mathscr{I}_\mathbf{1})\left\{\frac{\partial\mathrm{I}_p}{\partial\mathbf{U}}(\mathbf{1})\cdot\mathbf{E}_1\right\}\left\{\frac{\partial\mathrm{I}_k}{\partial\mathbf{U}}(\mathbf{1})\cdot\mathbf{E}_1\right\}\frac{\partial\mathrm{I}_j}{\partial\mathbf{U}}(\mathbf{1})\\ &\quad+2\frac{\partial^2\sigma}{\partial\mathrm{I}_k\,\partial\mathrm{I}_j}(\mathscr{I}_\mathbf{1})\left\{\frac{\partial^2\mathrm{I}_j}{\partial\mathbf{U}^2}(\mathbf{1})[\mathbf{E}_1]\right\}\left\{\frac{\partial\mathrm{I}_k}{\partial\mathbf{U}}(\mathbf{1})\cdot\mathbf{E}_1\right\}\\ &\quad+\frac{\partial^2\sigma}{\partial\mathrm{I}_k\,\partial\mathrm{I}_j}(\mathscr{I}_\mathbf{1})\frac{\partial\mathrm{I}_j}{\partial\mathbf{U}}(\mathbf{1})\left\{\mathbf{E}_1\cdot\frac{\partial^2\mathrm{I}_k}{\partial\mathbf{U}^2}(\mathbf{1})[\mathbf{E}_1]\right\}+\left\{\frac{\partial\sigma}{\partial\mathrm{I}_j}(\mathscr{I}_\mathbf{1})\right\}D\left\{\frac{\partial^2\mathrm{I}_j}{\partial\mathbf{U}^2}(\mathbf{1})[\mathbf{E}_1]\right\}[\mathbf{E}_1].\end{aligned}\tag{3.10}$$

Again, for compactness of notation, the name $\Psi_2[\mathbf{E}_1,\mathbf{E}_1]$ is assigned to $D^2(\partial\hat{\sigma}/\partial\mathbf{U})(\mathbf{1})[\mathbf{E}_1,\mathbf{E}_1]$.

Finally, (3.3) together with (3.4) gives the desired second order expansion for $(\partial\hat{\sigma}/\partial\mathbf{U})(\mathbf{U})$:

$$\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{U})=\Psi_1[\mathbf{E}_1]+\frac{1}{2}\Psi_2[\mathbf{E}_1,\mathbf{E}_1]+o(\mathbf{E}_1^2),\tag{3.11}$$

where $\Psi_1[\mathbf{E}_1]$ and $\Psi_2[\mathbf{E}_1,\mathbf{E}_1]$ are given by (3.7) and (3.10), respectively. Of course, $\Psi_1[\mathbf{E}_1]$ contains only terms that are first order in $\mathbf{E}_1$, and $\Psi_2[\mathbf{E}_1,\mathbf{E}_1]$ contains terms that are of order $\mathbf{E}_1^2$.

3.2. *The scalar coefficients*

Here the expressions required for the expansion of terms which are functions of the set {I, II, III} in stresses (2.28) and (2.27) to second order in the strain $\mathbf{E}_1$ are displayed. The invariants of $\mathbf{U}$ can be expressed in terms of the basic polynomial invariants of $\mathbf{E}_1$ as

$$\begin{aligned}\mathrm{I}&=\operatorname{tr}\mathbf{U}=3+\operatorname{tr}\mathbf{E}_1,\\ \mathrm{II}&=\tfrac{1}{2}((\operatorname{tr}\mathbf{U})^2-\operatorname{tr}\mathbf{U}^2)=3+2\operatorname{tr}\mathbf{E}_1+\tfrac{1}{2}((\operatorname{tr}\mathbf{E}_1)^2-\operatorname{tr}\mathbf{E}_1^2),\\ \mathrm{III}&=\tfrac{1}{6}((\operatorname{tr}\mathbf{U})^3-3(\operatorname{tr}\mathbf{U})(\operatorname{tr}\mathbf{U}^2)+2\operatorname{tr}\mathbf{U}^3)\\ &=1+\operatorname{tr}\mathbf{E}_1+\tfrac{1}{2}((\operatorname{tr}\mathbf{E}_1)^2-\operatorname{tr}\mathbf{E}_1^2)+\tfrac{1}{6}((\operatorname{tr}\mathbf{E}_1)^3-3(\operatorname{tr}\mathbf{E}_1)(\operatorname{tr}\mathbf{E}_1^2)+2\operatorname{tr}\mathbf{E}_1^3),\end{aligned}\tag{3.12}$$

which can be rewritten in terms of the principal invariants of $\mathbf{E}_1$ as

$$\begin{aligned}\mathrm{I}&=3+\mathrm{I}_{\mathbf{E}_1},\\ \mathrm{II}&=3+2\mathrm{I}_{\mathbf{E}_1}+\mathrm{II}_{\mathbf{E}_1},\\ \mathrm{III}&=1+\mathrm{I}_{\mathbf{E}_1}+\mathrm{II}_{\mathbf{E}_1}+\mathrm{III}_{\mathbf{E}_1}.\end{aligned}\tag{3.13}$$

Straightforward multiplication yields the following expressions for some of the scalar coefficients that appear in eqn (2.28):

$$\begin{aligned} \mathrm{I}^2 &= 9+6\mathrm{I}_{\mathbf{E}_1}+(\mathrm{I}_{\mathbf{E}_1})^2,\\ \mathrm{I}^2+\mathrm{II} &= 12+8\mathrm{I}_{\mathbf{E}_1}+((\mathrm{I}_{\mathbf{E}_1})^2+\mathrm{II}_{\mathbf{E}_1}),\\ \mathrm{I}\,\mathrm{III} &= 3+4\mathrm{I}_{\mathbf{E}_1}+(\mathrm{I}_{\mathbf{E}_1})^2+3\mathrm{II}_{\mathbf{E}_1}+o(\mathbf{E}_1^2),\\ \mathrm{I}\,\mathrm{II}-\mathrm{III} &= 8+8\mathrm{I}_{\mathbf{E}_1}+2((\mathrm{I}_{\mathbf{E}_1})^2+\mathrm{II}_{\mathbf{E}_1})+o(\mathbf{E}_1^2). \end{aligned} \tag{3.14}$$

The remaining scalar terms can each be expanded in a Taylor series, with the result

$$\begin{aligned} \mathrm{III}^{-1} &= 1-\mathrm{I}_{\mathbf{E}_1}+(\mathrm{I}_{\mathbf{E}_1})^2-\mathrm{II}_{\mathbf{E}_1}+o(\mathbf{E}_1^2),\\ (\mathrm{I}\,\mathrm{II}-\mathrm{III})^{-1} &= \tfrac{1}{8}(1-\mathrm{I}_{\mathbf{E}_1}+\mathrm{II}_{\mathbf{E}_1})+o(\mathbf{E}_1^2). \end{aligned} \tag{3.15}$$

3.3. *The stress*

Recall that (2.28) gives the Cauchy stress in terms of $(\partial\hat{\sigma}/\partial\mathbf{U})(\mathbf{U})$; evaluation of $\mathbf{U}$ at $\mathbf{1}+\mathbf{E}_1$, some algebra, and incorporation of the second order expressions for $(\partial\hat{\sigma}/\partial\mathbf{U})(\mathbf{U})$ given by (3.11) and eqns (3.14) and (3.15) for the scalar functions of the principal invariants provide the second order constitutive equation for Cauchy stress:

$$\mathbf{T}=\mathbf{R}\{\Psi_1[\mathbf{E}_1]+\mathrm{sym}(\mathbf{E}_1\Psi_1[\mathbf{E}_1])-(\mathrm{tr}\,\mathbf{E}_1)\Psi_1[\mathbf{E}_1]+\tfrac{1}{2}\Psi_2[\mathbf{E}_1,\mathbf{E}_1]\}\mathbf{R}^{\mathrm{T}}+o(\mathbf{E}_1^2), \tag{3.16}$$

with $\Psi_1[\mathbf{E}_1]$ and $\Psi_2[\mathbf{E}_1,\mathbf{E}_1]$ given by (3.7) and (3.10), respectively.

By (2.19) and (3.16), the second order constitutive equation for Piola–Kirchhoff stress is

$$\mathbf{S}=\mathbf{R}(\Psi_1[\mathbf{E}_1]+\mathrm{skw}(\mathbf{E}_1\Psi_1[\mathbf{E}_1])+\tfrac{1}{2}\Psi_2[\mathbf{E}_1,\mathbf{E}_1])+o(\mathbf{E}_1^2). \tag{3.17}$$

Equations (3.16) and (3.17) are the general second order constitutive equations for a hyperelastic material with arbitrary symmetry in terms of the Biot strain. Given an expression for the strain energy, an explicit expression can be obtained for a particular material, but no further information can be obtained for the general case. However, if the symmetry of the material is known, more can be done as will be shown in the next section.

4. Specific symmetries

In this section the forms of the second order stress constitutive equation derived in Section 3 are specialized for isotropic and transversely isotropic materials.

4.1. *Isotropy*

A material is isotropic at point $\mathbf{p}$ if the symmetry group at that point is the set of all rotations: $\mathscr{G}_{\mathbf{p}}=\{\mathrm{Orth}^+\}$. For isotropy the set of basic polynomial invariants of $\mathbf{U}$ is

$$\mathscr{I}_{\mathbf{U}} = \{\mathrm{I}_1(\mathbf{U}), \mathrm{I}_2(\mathbf{U}), \mathrm{I}_3(\mathbf{U})\} = \{\mathbf{1}\cdot\mathbf{U}, \mathbf{1}\cdot\mathbf{U}^2, \mathbf{1}\cdot\mathbf{U}^3\} \tag{4.1}$$[3]

and $\mathscr{I}_{\mathbf{1}} = \{3, 3, 3\}$. The derivatives of the basic polynomial invariants with respect to $\mathbf{U}$ are

$$\frac{\partial \mathrm{I}_1}{\partial \mathbf{U}} = \mathbf{1}, \quad \frac{\partial \mathrm{I}_2}{\partial \mathbf{U}} = 2\mathbf{U}, \quad \frac{\partial \mathrm{I}_3}{\partial \mathbf{U}} = 3\mathbf{U}^2. \tag{4.2}$$

Direct calculation gives the terms that appear in $\Psi_1[\mathbf{E}_1]$ and $\Psi_2[\mathbf{E}_1, \mathbf{E}_1]$ defined by eqns (3.7) and (3.10). The results are

$$\Psi_1[\mathbf{E}_1] = \eta_1(\mathbf{1}\cdot\mathbf{E}_1)\mathbf{1} + \eta_2\mathbf{E}_1 \tag{4.3}$$

and

$$\Psi_2[\mathbf{E}_1, \mathbf{E}_1] = 2\{\eta_5(\mathbf{1}\cdot\mathbf{E}_1)^2\mathbf{1} + \eta_4(\mathbf{1}\cdot\mathbf{E}_1^2)\mathbf{1} + 2\eta_4(\mathbf{1}\cdot\mathbf{E}_1)\mathbf{E}_1 + \eta_3\mathbf{E}_1^2\}, \tag{4.4}$$

with the constants η_i defined through

$$\eta_1 = \frac{\partial^2\sigma}{\partial \mathrm{I}_1\,\partial \mathrm{I}_1}(\mathscr{I}_{\mathbf{1}}) + 4\frac{\partial^2\sigma}{\partial \mathrm{I}_2\,\partial \mathrm{I}_2}(\mathscr{I}_{\mathbf{1}}) + 9\frac{\partial^2\sigma}{\partial \mathrm{I}_3\,\partial \mathrm{I}_3}(\mathscr{I}_{\mathbf{1}}) + 4\frac{\partial^2\sigma}{\partial \mathrm{I}_1\,\partial \mathrm{I}_2}(\mathscr{I}_{\mathbf{1}}) + 6\frac{\partial^2\sigma}{\partial \mathrm{I}_1\,\partial \mathrm{I}_3}(\mathscr{I}_{\mathbf{1}}) + 12\frac{\partial^2\sigma}{\partial \mathrm{I}_2\,\partial \mathrm{I}_3}(\mathscr{I}_{\mathbf{1}}), \tag{4.5}$$

$$\eta_2 = 2\frac{\partial\sigma}{\partial \mathrm{I}_2}(\mathscr{I}_{\mathbf{1}}) + 6\frac{\partial\sigma}{\partial \mathrm{I}_3}(\mathscr{I}_{\mathbf{1}}), \tag{4.6}$$

$$\eta_3 = 3\frac{\partial\sigma}{\partial \mathrm{I}_3}(\mathscr{I}_{\mathbf{1}}), \tag{4.7}$$

$$\eta_4 = 2\frac{\partial^2\sigma}{\partial \mathrm{I}_2\,\partial \mathrm{I}_2}(\mathscr{I}_{\mathbf{1}}) + 9\frac{\partial^2\sigma}{\partial \mathrm{I}_3\,\partial \mathrm{I}_3}(\mathscr{I}_{\mathbf{1}}) + \frac{\partial^2\sigma}{\partial \mathrm{I}_1\,\partial \mathrm{I}_2}(\mathscr{I}_{\mathbf{1}}) + 3\frac{\partial^2\sigma}{\partial \mathrm{I}_1\,\partial \mathrm{I}_3}(\mathscr{I}_{\mathbf{1}}) + 9\frac{\partial^2\sigma}{\partial \mathrm{I}_2\,\partial \mathrm{I}_3}(\mathscr{I}_{\mathbf{1}}), \tag{4.8}$$

$$\begin{aligned}\eta_5 = \frac{1}{2}\Bigg\{&\frac{\partial^3\sigma}{\partial \mathrm{I}_1\,\partial \mathrm{I}_1\,\partial \mathrm{I}_1}(\mathscr{I}_{\mathbf{1}}) + 6\frac{\partial^3\sigma}{\partial \mathrm{I}_1\,\partial \mathrm{I}_1\,\partial \mathrm{I}_2}(\mathscr{I}_{\mathbf{1}}) + 9\frac{\partial^3\sigma}{\partial \mathrm{I}_1\,\partial \mathrm{I}_1\,\partial \mathrm{I}_3}(\mathscr{I}_1)\\ &+ 12\frac{\partial^3\sigma}{\partial \mathrm{I}_1\,\partial \mathrm{I}_2\,\partial \mathrm{I}_2}(\mathscr{I}_{\mathbf{1}}) + 36\frac{\partial^3\sigma}{\partial \mathrm{I}_1\,\partial \mathrm{I}_2\,\partial \mathrm{I}_3}(\mathscr{I}_{\mathbf{1}}) + 27\frac{\partial^3\sigma}{\partial \mathrm{I}_1\,\partial \mathrm{I}_3\,\partial \mathrm{I}_3}(\mathscr{I}_{\mathbf{1}})\\ &+ 8\frac{\partial^3\sigma}{\partial \mathrm{I}_2\,\partial \mathrm{I}_2\,\partial \mathrm{I}_2}(\mathscr{I}_{\mathbf{1}}) + 36\frac{\partial^3\sigma}{\partial \mathrm{I}_2\,\partial \mathrm{I}_2\,\partial \mathrm{I}_3}(\mathscr{I}_{\mathbf{1}}) + 54\frac{\partial^3\sigma}{\partial \mathrm{I}_2\,\partial \mathrm{I}_3\,\partial \mathrm{I}_3}(\mathscr{I}_{\mathbf{1}}) + 27\frac{\partial^3\sigma}{\partial \mathrm{I}_3\,\partial \mathrm{I}_3\,\partial \mathrm{I}_3}(\mathscr{I}_{\mathbf{1}})\Bigg\}.\end{aligned} \tag{4.9}$$

Incorporation of these results into eqn (3.16) for the Cauchy stress leads to

[3] Alternatively, the invariants {I, II, III} defined in (2.9) could be used, but the set of basic polynomial invariants is more convenient for the calculations.

$$\mathbf{T} = \mathbf{R}\{\eta_1(\mathbf{1}\cdot\mathbf{E}_1)\mathbf{1}+\eta_2\mathbf{E}_1+(\eta_2+\eta_3)\mathbf{E}_1^2+\eta_4(\mathbf{1}\cdot\mathbf{E}_1^2)\mathbf{1} + (\eta_1-\eta_2+2\eta_4)(\mathbf{1}\cdot\mathbf{E}_1)\mathbf{E}_1+(\eta_5-\eta_1)(\mathbf{1}\cdot\mathbf{E}_1)^2\mathbf{1}\}\mathbf{R}^{\mathrm{T}}+o(\mathbf{E}_1^2). \quad (4.10)$$

This expression reduces to the standard first order constitutive equation for an isotropic hyperelastic material with two constants. Of course, given a particular isotropic strain energy function, it is straightforward to calculate the terms in (4.10), which would give the explicit second order isotropic hyperelastic constitutive equation. More typically, however, the values of the constants for a particular material would have to be obtained by experiment.

4.2. *Transverse isotropy*

A material is transversely isotropic at point **p** if the symmetry group at that point is the set of all rotations about a single preferred axis. Thus, the symmetry group for a transversely isotropic material at **p** is $\mathscr{G}_{\mathbf{p}} = \{\mathbf{Q}\in\mathrm{Orth}^+ : \mathbf{Qk}=\mathbf{k}\}$, where the unit vector **k** coincides with the axis of symmetry at **p**.

The complete list of basic polynomial invariants for transverse isotropy is

$$\begin{aligned}\mathscr{I}_{\mathbf{U}} &= \{\mathrm{I}_1(\mathbf{U}),\mathrm{I}_2(\mathbf{U}),\mathrm{I}_3(\mathbf{U}),\mathrm{I}_4(\mathbf{U}),\mathrm{I}_5(\mathbf{U})\}\\ &= \{\mathbf{1}\cdot\mathbf{U},\mathbf{1}\cdot\mathbf{U}^2,\mathbf{1}\cdot\mathbf{U}^3,\mathbf{k}\cdot\mathbf{Uk},\mathbf{k}\cdot\mathbf{U}^2\mathbf{k}\},\end{aligned} \quad (4.11)$$

and $\mathscr{I}_{\mathbf{1}} = \{3,3,3,1,1\}$. It will be convenient to write all of the invariants as inner products of tensors; to this end note that

$$\mathbf{k}\cdot(\mathbf{Ak}) = (\mathbf{k}\otimes\mathbf{k})\cdot\mathbf{A} = \mathbf{1}\cdot\{(\mathbf{k}\otimes\mathbf{k})\mathbf{A}\}$$

for all tensors **A**.

The derivatives of the basic polynomial invariants with respect to **U** are given by (4.2) and

$$\frac{\partial \mathrm{I}_4}{\partial \mathbf{U}} = (\mathbf{k}\otimes\mathbf{k}), \quad \frac{\partial \mathrm{I}_5}{\partial \mathbf{U}} = (\mathbf{k}\otimes\mathbf{k})\mathbf{U}+\mathbf{U}(\mathbf{k}\otimes\mathbf{k}). \quad (4.12)$$

Direct calculation together with (4.2) and (4.12) gives $\Psi_1[\mathbf{E}_1]$ from eqn (3.7), and $\Psi_2[\mathbf{E}_1,\mathbf{E}_1]$ from (3.10):

$$\Psi_1[\mathbf{E}_1] = \eta_1(\mathbf{1}\cdot\mathbf{E}_1)\mathbf{1}+\eta_2\mathbf{E}_1+\mu_1(\mathbf{1}\cdot\mathbf{E}_1)(\mathbf{k}\otimes\mathbf{k})+\mu_1[(\mathbf{k}\otimes\mathbf{k})\cdot\mathbf{E}_1]\mathbf{1} + \mu_2[(\mathbf{k}\otimes\mathbf{k})\cdot\mathbf{E}_1](\mathbf{k}\otimes\mathbf{k})+\mu_3[(\mathbf{k}\otimes\mathbf{k})\mathbf{E}_1+\mathbf{E}_1(\mathbf{k}\otimes\mathbf{k})]; \quad (4.13)$$

$$\begin{aligned}\Psi_2[\mathbf{E}_1,\mathbf{E}_1] = 2\{&\eta_3\mathbf{E}_1^2+\eta_4(\mathbf{1}\cdot\mathbf{E}_1^2)\mathbf{1}+2\eta_4(\mathbf{1}\cdot\mathbf{E}_1)\mathbf{E}_1+\eta_5(\mathbf{1}\cdot\mathbf{E}_1)^2\mathbf{1}\\ &+\mu_4\{(\mathbf{k}\otimes\mathbf{k})\cdot\mathbf{E}_1^2\}\mathbf{1}+\mu_4(\mathbf{1}\cdot\mathbf{E}_1)\{(\mathbf{k}\otimes\mathbf{k})\mathbf{E}_1+\mathbf{E}_1(\mathbf{k}\otimes\mathbf{k})\}\\ &+2\mu_5\{(\mathbf{k}\otimes\mathbf{k})\cdot\mathbf{E}_1\}\mathbf{E}_1+\mu_5(\mathbf{1}\cdot\mathbf{E}_1^2)(\mathbf{k}\otimes\mathbf{k})+\mu_6\{(\mathbf{k}\otimes\mathbf{k})\cdot\mathbf{E}_1^2\}(\mathbf{k}\otimes\mathbf{k})\\ &+\mu_6\{(\mathbf{k}\otimes\mathbf{k})\cdot\mathbf{E}_1\}\{(\mathbf{k}\otimes\mathbf{k})\mathbf{E}_1+\mathbf{E}_1(\mathbf{k}\otimes\mathbf{k})\}+\mu_7(\mathbf{1}\cdot\mathbf{E}_1)\{(\mathbf{k}\otimes\mathbf{k})\cdot\mathbf{E}_1\}\mathbf{1}\\ &+\mu_8\{(\mathbf{k}\otimes\mathbf{k})\cdot\mathbf{E}_1\}^2\mathbf{1}+2\mu_8(\mathbf{1}\cdot\mathbf{E}_1)\{(\mathbf{k}\otimes\mathbf{k})\cdot\mathbf{E}_1\}(\mathbf{k}\otimes\mathbf{k})\\ &+\mu_9(\mathbf{1}\cdot\mathbf{E}_1)^2(\mathbf{k}\otimes\mathbf{k})+\mu_{10}\{(\mathbf{k}\otimes\mathbf{k})\cdot\mathbf{E}_1\}^2(\mathbf{k}\otimes\mathbf{k})\};\end{aligned} \quad (4.14)$$

where the η_i are given in eqns (4.3)–(4.7), and the μ_i are defined by

$$\mu_1 = \frac{\partial^2\sigma}{\partial \mathrm{I}_1\,\partial \mathrm{I}_4}(\mathscr{I}_1) + 2\frac{\partial^2\sigma}{\partial \mathrm{I}_2\,\partial \mathrm{I}_4}(\mathscr{I}_1) + 3\frac{\partial^2\sigma}{\partial \mathrm{I}_3\,\partial \mathrm{I}_4}(\mathscr{I}_1) + 2\frac{\partial^2\sigma}{\partial \mathrm{I}_1\,\partial \mathrm{I}_5}(\mathscr{I}_1) + 4\frac{\partial^2\sigma}{\partial \mathrm{I}_2\,\partial \mathrm{I}_5}(\mathscr{I}_1) + 6\frac{\partial^2\sigma}{\partial \mathrm{I}_3\,\partial \mathrm{I}_5}(\mathscr{I}_1), \tag{4.15}$$

$$\mu_2 = \frac{\partial^2\sigma}{\partial \mathrm{I}_4\,\partial \mathrm{I}_4}(\mathscr{I}_1) + 4\frac{\partial^2\sigma}{\partial \mathrm{I}_4\,\partial \mathrm{I}_5}(\mathscr{I}_1) + 4\frac{\partial^2\sigma}{\partial \mathrm{I}_5\,\partial \mathrm{I}_5}(\mathscr{I}_1), \tag{4.16}$$

$$\mu_3 = \frac{\partial\sigma}{\partial \mathrm{I}_5}(\mathscr{I}_1), \tag{4.17}$$

$$\mu_4 = \frac{\partial^2\sigma}{\partial \mathrm{I}_1\,\partial \mathrm{I}_5}(\mathscr{I}_1) + 2\frac{\partial^2\sigma}{\partial \mathrm{I}_2\,\partial \mathrm{I}_5}(\mathscr{I}_1) + 3\frac{\partial^2\sigma}{\partial \mathrm{I}_3\,\partial \mathrm{I}_5}(\mathscr{I}_1), \tag{4.18}$$

$$\mu_5 = \frac{\partial^2\sigma}{\partial \mathrm{I}_2\,\partial \mathrm{I}_4}(\mathscr{I}_1) + 3\frac{\partial^2\sigma}{\partial \mathrm{I}_3\,\partial \mathrm{I}_4}(\mathscr{I}_1) + 2\frac{\partial^2\sigma}{\partial \mathrm{I}_2\,\partial \mathrm{I}_5}(\mathscr{I}_1) + 6\frac{\partial^2\sigma}{\partial \mathrm{I}_3\,\partial \mathrm{I}_5}(\mathscr{I}_1), \tag{4.19}$$

$$\mu_6 = \frac{\partial^2\sigma}{\partial \mathrm{I}_4\,\partial \mathrm{I}_5}(\mathscr{I}_1) + 2\frac{\partial^2\sigma}{\partial \mathrm{I}_5\,\partial \mathrm{I}_5}(\mathscr{I}_1), \tag{4.20}$$

$$\begin{aligned}\mu_7 = {} & \frac{\partial^3\sigma}{\partial \mathrm{I}_1\,\partial \mathrm{I}_1\,\partial \mathrm{I}_4}(\mathscr{I}_1) + 2\frac{\partial^3\sigma}{\partial \mathrm{I}_1\,\partial \mathrm{I}_1\,\partial \mathrm{I}_5}(\mathscr{I}_1) + 4\frac{\partial^3\sigma}{\partial \mathrm{I}_1\,\partial \mathrm{I}_2\,\partial \mathrm{I}_4}(\mathscr{I}_1) \\ & + 8\frac{\partial^3\sigma}{\partial \mathrm{I}_1\,\partial \mathrm{I}_2\,\partial \mathrm{I}_5}(\mathscr{I}_1) + 6\frac{\partial^3\sigma}{\partial \mathrm{I}_1\,\partial \mathrm{I}_3\,\partial \mathrm{I}_4}(\mathscr{I}_1) + 12\frac{\partial^3\sigma}{\partial \mathrm{I}_1\,\partial \mathrm{I}_3\,\partial \mathrm{I}_5}(\mathscr{I}_1) \\ & + 4\frac{\partial^3\sigma}{\partial \mathrm{I}_2\,\partial \mathrm{I}_2\,\partial \mathrm{I}_4}(\mathscr{I}_1) + 8\frac{\partial^3\sigma}{\partial \mathrm{I}_2\,\partial \mathrm{I}_2\,\partial \mathrm{I}_5}(\mathscr{I}_1) + 12\frac{\partial^3\sigma}{\partial \mathrm{I}_2\,\partial \mathrm{I}_3\,\partial \mathrm{I}_4}(\mathscr{I}_1) + 24\frac{\partial^3\sigma}{\partial \mathrm{I}_2\,\partial \mathrm{I}_3\,\partial \mathrm{I}_5}(\mathscr{I}_1) \\ & + 9\frac{\partial^3\sigma}{\partial \mathrm{I}_3\,\partial \mathrm{I}_3\,\partial \mathrm{I}_4}(\mathscr{I}_1) + 18\frac{\partial^3\sigma}{\partial \mathrm{I}_3\,\partial \mathrm{I}_3\,\partial \mathrm{I}_5}(\mathscr{I}_1),\end{aligned} \tag{4.21}$$

$$\begin{aligned}\mu_8 = {} & \frac{1}{2}\left\{\frac{\partial^3\sigma}{\partial \mathrm{I}_1\,\partial \mathrm{I}_4\,\partial \mathrm{I}_4}(\mathscr{I}_1) + 4\frac{\partial^3\sigma}{\partial \mathrm{I}_1\,\partial \mathrm{I}_4\,\partial \mathrm{I}_5}(\mathscr{I}_1) + 4\frac{\partial^3\sigma}{\partial \mathrm{I}_1\,\partial \mathrm{I}_5\,\partial \mathrm{I}_5}(\mathscr{I}_1)\right. \\ & + 2\frac{\partial^3\sigma}{\partial \mathrm{I}_2\,\partial \mathrm{I}_4\,\partial \mathrm{I}_4}(\mathscr{I}_1) + 8\frac{\partial^3\sigma}{\partial \mathrm{I}_2\,\partial \mathrm{I}_4\,\partial \mathrm{I}_5}(\mathscr{I}_1) + 8\frac{\partial^3\sigma}{\partial \mathrm{I}_2\,\partial \mathrm{I}_5\,\partial \mathrm{I}_5}(\mathscr{I}_1) \\ & \left. + 3\frac{\partial^3\sigma}{\partial \mathrm{I}_3\,\partial \mathrm{I}_4\,\partial \mathrm{I}_4}(\mathscr{I}_1) + 12\frac{\partial^3\sigma}{\partial \mathrm{I}_3\,\partial \mathrm{I}_4\,\partial \mathrm{I}_5}(\mathscr{I}_1) + 12\frac{\partial^3\sigma}{\partial \mathrm{I}_3\,\partial \mathrm{I}_5\,\partial \mathrm{I}_5}(\mathscr{I}_1)\right\},\end{aligned} \tag{4.22}$$

$$\mu_9 = \frac{1}{2}\left\{\frac{\partial^3\sigma}{\partial \mathrm{I}_1\,\partial \mathrm{I}_1\,\partial \mathrm{I}_4}(\mathscr{I}_1) + 4\frac{\partial^3\sigma}{\partial \mathrm{I}_1\,\partial \mathrm{I}_2\,\partial \mathrm{I}_4}(\mathscr{I}_1) + 6\frac{\partial^3\sigma}{\partial \mathrm{I}_1\,\partial \mathrm{I}_3\,\partial \mathrm{I}_4}(\mathscr{I}_1)\right.$$

$$+4\frac{\partial^3\sigma}{\partial I_2\,\partial I_2\,\partial I_4}(\mathscr{I}_1)+12\frac{\partial^3\sigma}{\partial I_2\,\partial I_3\,\partial I_4}(\mathscr{I}_1)+9\frac{\partial^3\sigma}{\partial I_3\,\partial I_3\,\partial I_4}(\mathscr{I}_1)$$
$$+2\frac{\partial^3\sigma}{\partial I_1\,\partial I_1\,\partial I_5}(\mathscr{I}_1)+8\frac{\partial^3\sigma}{\partial I_1\,\partial I_2\,\partial I_5}(\mathscr{I}_1)+12\frac{\partial^3\sigma}{\partial I_1\,\partial I_3\,\partial I_5}(\mathscr{I}_1)$$
$$+8\frac{\partial^3\sigma}{\partial I_2\,\partial I_2\,\partial I_5}(\mathscr{I}_1)+24\frac{\partial^3\sigma}{\partial I_2\,\partial I_3\,\partial I_5}(\mathscr{I}_1)+18\frac{\partial^3\sigma}{\partial I_3\,\partial I_3\,\partial I_5}(\mathscr{I}_1)\Big\}, \tag{4.23}$$

$$\mu_{10}=\frac{1}{2}\left\{\frac{\partial^3\sigma}{\partial I_4\,\partial I_4\,\partial I_4}(\mathscr{I}_1)+6\frac{\partial^3\sigma}{\partial I_4\,\partial I_4\,\partial I_5}(\mathscr{I}_1)+12\frac{\partial^3\sigma}{\partial I_4\,\partial I_5\,\partial I_5}(\mathscr{I}_1)+8\frac{\partial^3\sigma}{\partial I_5\,\partial I_5\,\partial I_5}(\mathscr{I}_1)\right\}. \tag{4.24}$$

These expressions for $\Psi_1[\mathbf{E}_1]$ and $\Psi_2[\mathbf{E}_1,\mathbf{E}_1]$, the identity

$$\begin{aligned}\mathbf{0}&=\mathbf{E}_1(\mathbf{k}\otimes\mathbf{k})\mathbf{E}_1+[(\mathbf{k}\otimes\mathbf{k})\mathbf{E}_1^2+\mathbf{E}_1^2(\mathbf{k}\otimes\mathbf{k})]\\&\quad-(\mathbf{1}\cdot\mathbf{E}_1)[(\mathbf{k}\otimes\mathbf{k})\mathbf{E}_1+\mathbf{E}_1(\mathbf{k}\otimes\mathbf{k})]-\mathbf{E}_1^2\\&\quad-[(\mathbf{k}\otimes\mathbf{k})\cdot\mathbf{E}_1]\mathbf{E}_1+(\mathbf{1}\cdot\mathbf{E}_1)\mathbf{E}_1+\tfrac{1}{2}[(\mathbf{1}\cdot\mathbf{E}_1)^2+(\mathbf{1}\cdot\mathbf{E}_1^2)](\mathbf{k}\otimes\mathbf{k})\\&\quad-\{[(\mathbf{k}\otimes\mathbf{k})\cdot\mathbf{E}_1^2]-(\mathbf{1}\cdot\mathbf{E}_1)[(\mathbf{k}\otimes\mathbf{k})\cdot\mathbf{E}_1]+\tfrac{1}{2}[(\mathbf{1}\cdot\mathbf{E}_1)^2+(\mathbf{1}\cdot\mathbf{E}_1^2)]\}\mathbf{1}\end{aligned} \tag{4.25}$$

(Rivlin, 1955) and eqn (3.17) for the Cauchy stress yield the second order constitutive equation for a transversely isotropic hyperelastic material:

$$\begin{aligned}\mathbf{T}&=\mathbf{R}\{\eta_1(\mathbf{1}\cdot\mathbf{E}_1)\mathbf{1}+\eta_2\mathbf{E}_1+\mu_1(\mathbf{1}\cdot\mathbf{E}_1)(\mathbf{k}\otimes\mathbf{k})+\mu_1[(\mathbf{k}\otimes\mathbf{k})\cdot\mathbf{E}_1]\mathbf{1}\\&\quad+\mu_2[(\mathbf{k}\otimes\mathbf{k})\cdot\mathbf{E}_1](\mathbf{k}\otimes\mathbf{k})+\mu_3[(\mathbf{k}\otimes\mathbf{k})\mathbf{E}_1+\mathbf{E}_1(\mathbf{k}\otimes\mathbf{k})]\\&\quad+(\eta_2+\eta_3+\mu_3)\mathbf{E}_1^2+(\eta_4-\tfrac{1}{2}\mu_3)(1\cdot\mathbf{E}_1^2)\mathbf{1}\\&\quad+(\eta_1-\eta_2+2\eta_4-\mu_3)(\mathbf{1}\cdot\mathbf{E}_1)\mathbf{E}_1+(\eta_5-\eta_1+\tfrac{1}{2}\mu_3)(\mathbf{1}\cdot\mathbf{E}_1)^2\mathbf{1}\\&\quad+(\tfrac{1}{2}\mu_1+\mu_4)(\mathbf{1}\cdot\mathbf{E}_1)\{(\mathbf{k}\otimes\mathbf{k})\mathbf{E}_1+\mathbf{E}_1(\mathbf{k}\otimes\mathbf{k})\}+(\mu_1+\mu_3+2\mu_5)\{(\mathbf{k}\otimes\mathbf{k})\cdot\mathbf{E}_1\}\mathbf{E}_1\\&\quad+(\tfrac{1}{2}\mu_2+\mu_6)\{(\mathbf{k}\otimes\mathbf{k})\cdot\mathbf{E}_1\}\{(\mathbf{k}\otimes\mathbf{k})\mathbf{E}_1+\mathbf{E}_1(\mathbf{k}\otimes\mathbf{k})\}\\&\quad-\mu_3\{(\mathbf{k}\otimes\mathbf{k})\mathbf{E}_1^2+\mathbf{E}_1^2(\mathbf{k}\otimes\mathbf{k})\}+(\mu_3+\mu_4)\{(\mathbf{k}\otimes\mathbf{k})\cdot\mathbf{E}_1^2\}\mathbf{1}\\&\quad+(-\mu_1-\tfrac{1}{2}\mu_3+\mu_9)(\mathbf{1}\cdot\mathbf{E}_1)^2(\mathbf{k}\otimes\mathbf{k})+(\tfrac{1}{2}\mu_3+\mu_5)(\mathbf{1}\cdot\mathbf{E}_1^2)(\mathbf{k}\otimes\mathbf{k})\\&\quad+(-\mu_1-\mu_3+\mu_7)(\mathbf{1}\cdot\mathbf{E}_1)\{(\mathbf{k}\otimes\mathbf{k})\cdot\mathbf{E}_1\}\mathbf{1}+\mu_6\{(\mathbf{k}\otimes\mathbf{k})\cdot\mathbf{E}_1^2\}(\mathbf{k}\otimes\mathbf{k})\\&\quad+(-\mu_2+2\mu_8)(\mathbf{1}\cdot\mathbf{E}_1)\{(\mathbf{k}\otimes\mathbf{k})\cdot\mathbf{E}_1\}(\mathbf{k}\otimes\mathbf{k})+\mu_8\{(\mathbf{k}\otimes\mathbf{k})\cdot\mathbf{E}_1\}^2\mathbf{1}\\&\quad+\mu_{10}\{(\mathbf{k}\otimes\mathbf{k})\cdot\mathbf{E}_1\}^2(\mathbf{k}\otimes\mathbf{k})\}\mathbf{R}^{\mathrm{T}}+o(\mathbf{E}_1^2).\end{aligned} \tag{4.26}$$

This expression reduces to the standard first order constitutive equation for a transversely isotropic hyperelastic material with five constants.

5. Other second order theories

The focus of this paper has been on the development of constitutive equations which are second order in the Biot strain $\mathbf{E}_1$. As discussed in Section 2.3, this strain measure was chosen due to its transparent relation to the physical quantity of extension. In this section, constitutive equations written in terms of two other deformation measures are considered. First, I will focus on the second order constitutive equation in $\mathbf{H}$. The section then closes with a discussion of a constitutive equation which is second order in the Green strain $\mathbf{E}_2$.

5.1. *Constitutive equations that are second order in* $\boldsymbol{H}$

A number of authors have obtained second order constitutive equations for isotropic elastic materials in which the stress is second order in the displacement gradient $\mathbf{H}$. An essential difficulty of these second order constitutive equations is that the strain and the rotation associated with the deformation cannot be separated; and because the rotation remains buried in the constitutive equation, it is difficult to construct an expression that is frame indifferent. This difficulty is not alleviated by use of the elongation tensor $\mathbf{E} = \operatorname{sym}\mathbf{H}$, since it is not a strain measure, but rather a simple linear combination of the displacement gradient (for a discussion see, e.g., Truesdell and Toupin, 1960). Despite this inherent difficulty, there are situations in which a constitutive equation that is second order in $\mathbf{H}$ may be useful.

This section begins with a derivation of the constitutive equation for a material with arbitrary symmetry that is second order in $\mathbf{H}$, and is formulated in terms of the derivatives of the strain energy function $\hat{\sigma}(\mathbf{U})$. I present my own derivation, using the same notation as Section 3 in terms of Biot strain, to make explicit the relation between constitutive equations derived in terms of these two different measures of deformation. This constitutive equation is then used to obtain the constitutive equation relevant to the special case of isotropy, and the result is compared with those of several standard references for constitutive equations which are second order in $\mathbf{H}$.

Derivation of the constitutive equation that is second order in $\boldsymbol{H}$

Recall from Section 2.5, eqn (2.24), that the Piola–Kirchhoff stress can be expressed as the gradient of the strain energy function $\bar{\sigma}$, and that $\hat{\sigma}$ was defined as the restriction of $\bar{\sigma}$ to the set of positive definite symmetric tensors.

First, note that $\mathbf{U}$ can be written in terms of $\mathbf{F}$ through

$$\mathbf{U} = \mathscr{U}(\mathbf{F}) = \sqrt{\mathbf{F}^{\mathrm{T}}\mathbf{F}}. \tag{5.1}$$

Thus, by 2.26,

$$\bar{\sigma}(\mathbf{F}) = \hat{\sigma}(\mathbf{U}) = \hat{\sigma}(\mathscr{U}(\mathbf{F})). \tag{5.2}$$

By the chain rule,

$$D\bar{\sigma}(\mathbf{F})[\mathbf{A}] = D\hat{\sigma}(\mathscr{U}(\mathbf{F}))[D\mathscr{U}(\mathbf{F})[\mathbf{A}]],$$

which can be rewritten in gradient notation as

$$\mathbf{A}\cdot\nabla\bar{\sigma}(\mathbf{F}) = \mathbf{A}\cdot\{D\mathscr{U}(\mathbf{F})\}^{\mathrm{T}}[\nabla\hat{\sigma}(\mathscr{U}(\mathbf{F}))]. \tag{5.3}$$

The increment $\mathbf{A}$ is an arbitrary constant tensor. Note that $D\mathscr{U}(\mathbf{F})$ is a fourth order tensor with components

$$\{D\mathscr{U}(\mathbf{F})\}_{ijkl} = \frac{\partial \mathscr{U}_{kl}}{\partial F_{ij}}(\mathbf{F}). \tag{5.4}$$

The goal here is to write $\nabla\bar{\sigma}(\mathbf{F})$ as a Taylor series in terms of the displacement gradient $\mathbf{H}$ and the constants displayed in Section 4.1. With $\mathbf{F} = \mathbf{1}+\mathbf{H}$, the gradient of the strain energy can be expanded as

$$\begin{aligned}\nabla\bar{\sigma}(\mathbf{F}) &= \nabla\bar{\sigma}(\mathbf{1}+\mathbf{H})\\ &= \nabla\bar{\sigma}(\mathbf{1}) + D\nabla\bar{\sigma}(\mathbf{1})[\mathbf{H}] + \tfrac{1}{2}D^2\nabla\bar{\sigma}(\mathbf{1})[\mathbf{H},\mathbf{H}] + o(\mathbf{H}^2)\end{aligned} \tag{5.5}$$

as $\mathbf{H}\to\mathbf{0}$.

Equation (5.3) evaluated at $\mathbf{F} = \mathbf{1}$, together with $\mathscr{U}(\mathbf{1}) = \mathbf{1}$ and recognition that $\mathbf{A}$ is arbitrary, give

$$\nabla\bar{\sigma}(\mathbf{1}) = \{D\mathscr{U}(\mathbf{1})\}^{\mathrm{T}}[\nabla\hat{\sigma}(\mathbf{1})].$$

Since $\nabla\hat{\sigma}(\mathbf{1}) = 0$ by (3.4),

$$\nabla\bar{\sigma}(\mathbf{1}) = 0. \tag{5.6}$$

Now consider $D\nabla\bar{\sigma}(\mathbf{1})[\mathbf{H}]$. By differentiating (5.3) one can obtain

$$D\nabla\bar{\sigma}(\mathbf{F})[\mathbf{H}]\cdot\mathbf{A} = D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathscr{U}(\mathbf{F}))[D\mathscr{U}(\mathbf{F})[\mathbf{H}]]\cdot D\mathscr{U}(\mathbf{F})[\mathbf{A}] + \frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathscr{U}(\mathbf{F}))\cdot D^2\mathscr{U}(\mathbf{F})[\mathbf{A},\mathbf{H}]. \tag{5.7}$$

Because $\mathbf{A}$ is arbitrary, this expression evaluated at $\mathbf{F} = \mathbf{1}$ together with (3.4) give

$$D\nabla\bar{\sigma}(\mathbf{1})[\mathbf{H}] = \{D\mathscr{U}(\mathbf{1})\}^{\mathrm{T}}\left[D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[D\mathscr{U}(\mathbf{1})[\mathbf{H}]]\right]. \tag{5.8}$$

To get any farther it is necessary to incorporate the explicit expression for the derivative of the function $\mathscr{U}(\mathbf{F}) = \sqrt{\mathbf{F}^{\mathrm{T}}\mathbf{F}}$. The required derivative can be shown to be

$$D\mathscr{U}(\mathbf{1})[\mathbf{A}] = \operatorname{sym}\mathbf{A}, \tag{5.9}$$

and, specifically, when the increment is $\mathbf{H}$,

$$D\mathscr{U}(\mathbf{1})[\mathbf{H}] = \mathbf{E}. \tag{5.10}$$

Thus (5.8) is just

$$D\nabla\bar{\sigma}(\mathbf{1})[\mathbf{H}] = D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[\mathbf{E}] \tag{5.11}$$

since the derivative of $\partial\hat{\sigma}/\partial\mathbf{U}$ is symmetric.

Now consider $D^2\nabla\bar{\sigma}(\mathbf{1})[\mathbf{H},\mathbf{H}]$. By (2.4) and the chain rule, the derivative of (5.7) is

$$\begin{aligned} D^2\nabla\bar{\sigma}(\mathscr{U}(\mathbf{F}))[\mathbf{H},\mathbf{H}]\cdot\mathbf{A} &= D(D\nabla\bar{\sigma}(\mathscr{U}(\mathbf{F}))[\mathbf{H}])[\mathbf{H}]\cdot\mathbf{A} \\ &= \left\{D\left(D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathscr{U}(\mathbf{F}))[D\mathscr{U}(\mathbf{F})[\mathbf{H}]]\right)[\mathbf{H}]\right\}\cdot D\mathscr{U}(\mathbf{F})[\mathbf{A}] \\ &\quad + 2D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathscr{U}(\mathbf{F}))[D\mathscr{U}(\mathbf{F})[\mathbf{H}]]\cdot D^2\mathscr{U}(\mathbf{F})[\mathbf{A},\mathbf{H}] \\ &\quad + \frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathscr{U}(\mathbf{F}))\cdot D(D^2\mathscr{U}(\mathbf{F})[\mathbf{A},\mathbf{H}])[\mathbf{H}]. \end{aligned} \tag{5.12}$$

The first part of the leading term on the right hand side of (5.12) can be obtained by use of the chain rule (2.6). The result, when evaluated at $\mathbf{F}=\mathbf{1}$, is

$$\begin{aligned} &\left\{D\left(D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[D\mathscr{U}(\mathbf{1})[\mathbf{H}]]\right)[\mathbf{H}]\right\} \\ &\qquad = D^2\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[D\mathscr{U}(\mathbf{1})[\mathbf{H}],D\mathscr{U}(\mathbf{1})[\mathbf{H}]] + D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[D^2\mathscr{U}(\mathbf{1})[\mathbf{H},\mathbf{H}]]. \end{aligned}$$

Recall that $(\partial\hat{\sigma}/\partial\mathbf{U})(\mathbf{1})=\mathbf{0}$, so the last term in (5.12) will vanish when evaluated at $\mathbf{F}=\mathbf{1}$. Thus,

$$\begin{aligned} D^2\nabla\bar{\sigma}(\mathscr{U}(\mathbf{1}))[\mathbf{H},\mathbf{H}]\cdot\mathbf{A} &= D^2\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[D\mathscr{U}(\mathbf{1})[\mathbf{H}],D\mathscr{U}(\mathbf{1})[\mathbf{H}]]\cdot D\mathscr{U}(\mathbf{1})[\mathbf{A}] \\ &\quad + D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[D^2\mathscr{U}(\mathbf{1})[\mathbf{H},\mathbf{H}]]\cdot D\mathscr{U}(\mathbf{1})[\mathbf{A}] + 2D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[D\mathscr{U}(\mathbf{1})[\mathbf{H}]]\cdot D^2\mathscr{U}(\mathbf{1})[\mathbf{A},\mathbf{H}]. \end{aligned} \tag{5.13}$$

To proceed the second derivatives of $\mathscr{U}(\mathbf{F})$ are required. These are

$$D^2\mathscr{U}(\mathbf{1})[\mathbf{A},\mathbf{H}] = \tfrac{1}{2}(\mathbf{A}^{\mathrm{T}}\mathbf{H}+\mathbf{H}^{\mathrm{T}}\mathbf{A}) - \tfrac{1}{2}(\mathbf{E}(\operatorname{sym}\mathbf{A}) + (\operatorname{sym}\mathbf{A})\mathbf{E}), \tag{5.14}$$

$$D^2\mathscr{U}(\mathbf{1})[\mathbf{H},\mathbf{H}] = \mathbf{H}^{\mathrm{T}}\mathbf{H} - \mathbf{E}^2. \tag{5.15}$$

The first and second derivative of $(\partial\hat{\sigma}/\partial\mathbf{U})(\mathbf{U})$ are symmetric and $\mathbf{A}$ is arbitrary, so eqns (5.9) and (5.10), together with a considerable amount of algebra, provide the sought for expression:

$$\begin{aligned} D^2\nabla\bar{\sigma}(\mathscr{U}(\mathbf{1}))[\mathbf{H},\mathbf{H}] &= D^2\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[\mathbf{E},\mathbf{E}] + D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[\mathbf{H}^{\mathrm{T}}\mathbf{H}-\mathbf{E}^2] \\ &\quad + 2\mathbf{H}\left(D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[\mathbf{E}]\right) + \left(D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[\mathbf{E}]\right)\mathbf{E} - \mathbf{E}\left(D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[\mathbf{E}]\right). \end{aligned} \tag{5.16}$$

Finally, (5.6), (5.11) and (5.16) can be incorporated into the Taylor expansion (5.5) to provide an expression for $\nabla\bar{\sigma}(\mathbf{F})$:

$$\nabla\bar{\sigma}(\mathbf{F}) = D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[\mathbf{E}] + \frac{1}{2}\left\{D^2\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[\mathbf{E},\mathbf{E}] + D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[\mathbf{H}^{\mathrm{T}}\mathbf{H}-\mathbf{E}^2]\right.$$
$$\left. + 2\mathbf{H}\left(D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[\mathbf{E}]\right) + \left(D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[\mathbf{E}]\right)\mathbf{E} - \mathbf{E}\left(D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[\mathbf{E}]\right)\right\} + o(\mathbf{E}^2). \quad (5.17)$$

Equations (2.24) and (2.19), together with the approximation $(\det\mathbf{F})^{-1} = 1-(\mathbf{1}\cdot\mathbf{E})+o(\mathbf{H})$, give the following formula for the Cauchy stress in terms of $\nabla\bar{\sigma}(\mathbf{F})$:

$$\mathbf{T} = \{1-(\mathbf{1}\cdot\mathbf{E})\}\nabla\bar{\sigma}(\mathbf{F})(\mathbf{1}+\mathbf{H}^{\mathrm{T}}) + o(\mathbf{E}^2). \quad (5.18)$$

Thus, by (5.17) and (5.18), the constitutive equation for a hyperelastic material of arbitrary symmetry that is second order in $\mathbf{H}$ can be written as

$$\mathbf{T} = D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[\mathbf{E}] - (\mathbf{1}\cdot\mathbf{E})\left(D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[\mathbf{E}]\right)\mathbf{1} + \left(D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[\mathbf{E}]\right)\mathbf{H}^{\mathrm{T}}$$
$$+ \frac{1}{2}\left\{D^2\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[\mathbf{E},\mathbf{E}] + D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[\mathbf{H}^{\mathrm{T}}\mathbf{H}-\mathbf{E}^2]\right.$$
$$\left. + 2\mathbf{H}\left(D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[\mathbf{E}]\right) - \left(D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[\mathbf{E}]\right)\mathbf{E} + \mathbf{E}\left(D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[\mathbf{E}]\right)\right\} + o(\mathbf{E}^2), \quad (5.19)$$

where the derivatives $D(\partial\hat{\sigma}/\partial\mathbf{U})(\mathbf{1})$ and $D^2(\partial\hat{\sigma}/\partial\mathbf{U})(\mathbf{1})$ are given by (3.7) and (3.10).

The second order constitutive equation for isotropic materials

Turning to the special case of isotropy recall that, for any increment $\mathbf{A}$,

$$\Psi_1[\mathbf{A}] = D\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[\mathbf{A}] = \eta_1(\mathbf{1}\cdot\mathbf{A}) + \eta_2\mathbf{A}$$

and

$$\frac{1}{2}\Psi_2[\mathbf{A},\mathbf{A}] = \frac{1}{2}D^2\frac{\partial\hat{\sigma}}{\partial\mathbf{U}}(\mathbf{1})[\mathbf{A},\mathbf{A}] = \eta_5(\mathbf{1}\cdot\mathbf{A})^2\mathbf{1} + \eta_4(\mathbf{1}\cdot\mathbf{A}^2)\mathbf{1} + 2\eta_4(\mathbf{1}\cdot\mathbf{A})\mathbf{A} + \eta_3\mathbf{A}^2$$

by (4.3) and (4.4), respectively, and the scalar coefficients are given by eqns (4.5) through (4.9). So for isotropy (5.19) becomes

$$\mathbf{T} = \eta_1(\mathbf{1}\cdot\mathbf{E})\mathbf{1} + \eta_2\mathbf{E} + (\eta_5-\eta_1)(\mathbf{1}\cdot\mathbf{E})^2\mathbf{1} + (\eta_4-\tfrac{1}{2}\eta_1)(\mathbf{1}\cdot\mathbf{E}^2)\mathbf{1}$$
$$+ (2\eta_4-\eta_2+\eta_1)(\mathbf{1}\cdot\mathbf{E})\mathbf{E} + (\eta_3-\tfrac{3}{2}\eta_2)\mathbf{E}^2 + \tfrac{1}{2}\eta_1(\mathbf{1}\cdot\mathbf{H}^{\mathrm{T}}\mathbf{H})\mathbf{1} + \eta_2(\mathbf{H}\mathbf{E}+\mathbf{E}\mathbf{H}^{\mathrm{T}}+\tfrac{1}{2}\mathbf{H}^{\mathrm{T}}\mathbf{H}) + o(\mathbf{E}^2). \quad (5.20)$$

Various authors have derived constitutive equations that are second order in $\mathbf{H}$ for hyperelastic isotropic materials, including Haughton and Lindsay (1993, 1994), Rivlin (1953), Murnaghan (1937), and Toupin and Bernstein (1961). Several of these constitutive equations are summarized in Section 66 of Truesdell and Noll (1965). The equation displayed there for Cauchy stress is[4]

$$\mathbf{T} = \beta_1(\mathbf{1}\cdot\mathbf{E})\mathbf{1} + 2\beta_2\mathbf{E} + \tfrac{1}{2}\beta_1(\mathbf{1}\cdot\mathbf{H}\mathbf{H}^{\mathrm{T}})\mathbf{1} + \beta_2\mathbf{H}\mathbf{H}^{\mathrm{T}} + \beta_3(\mathbf{1}\cdot\mathbf{E})^2 + (2\beta_1 - 2\beta_2 - \beta_4)\tfrac{1}{2}\{(\mathbf{1}\cdot\mathbf{E})^2 - (\mathbf{1}\cdot\mathbf{E}^2)\} + \beta_4(\mathbf{1}\cdot\mathbf{E})\mathbf{E} + \beta_5\mathbf{E}^2 + o(\mathbf{E}^2). \quad (5.21)$$

By noting that

$$\mathbf{H}\mathbf{E} + \mathbf{E}\mathbf{H}^{\mathrm{T}} + \tfrac{1}{2}\mathbf{H}^{\mathrm{T}}\mathbf{H} = 2\mathbf{E}^2 + \tfrac{1}{2}\mathbf{H}\mathbf{H}^{\mathrm{T}}$$

and

$$\mathbf{1}\cdot\mathbf{H}\mathbf{H}^{\mathrm{T}} = \mathbf{1}\cdot\mathbf{H}^{\mathrm{T}}\mathbf{H},$$

one can easily show that (5.20) is equivalent to (5.21) where the constants are related through

$$\beta_1 = \eta_1, \quad \beta_2 = \tfrac{1}{2}\eta_2, \quad \beta_3 = \eta_5 + \eta_4 - \tfrac{3}{2}\eta_1, \quad \beta_4 = 2\eta_4 + \eta_1 - \eta_2, \quad \text{and} \quad \beta_5 = \eta_3 + \tfrac{1}{2}\eta_2.$$

5.2. *A constitutive equation that is second order in* $\boldsymbol{E}_2$

Murnaghan (1951) obtained a second order constitutive equation by approximating the strain energy function of an isotropic material as a sum of terms which were, respectively, zeroth, first, second and third order in the Green strain $\mathbf{E}_2$. By differentiating this approximation of the strain energy, and incorporating the result into (3.17), he obtained

$$\mathbf{T} = \mathbf{R}\{\lambda(\mathbf{1}\cdot\mathbf{E}_2)\mathbf{1} + 2\mu\mathbf{E}_2 + (4\mu + n)\mathbf{E}_2^2 + (n - 2m)\tfrac{1}{2}[(\mathbf{1}\cdot\mathbf{E}_2)^2 + (\mathbf{1}\cdot\mathbf{E}_2^2)]\mathbf{1} + (2\lambda - 2\mu + 2m - n)(\mathbf{1}\cdot\mathbf{E}_2)\mathbf{E}_2 + (l - \lambda)(\mathbf{1}\cdot\mathbf{E}_2)^2\mathbf{1}\}\mathbf{R}^{\mathrm{T}} + o(\mathbf{E}_2^2), \quad (5.22)$$

where λ and μ are the classical first order elastic constants, and l, m, and n are Murnaghan's second order elastic constants. Recall that $\mathbf{E}_2$ is second order in the principal extensions, as discussed in Section 2.3, so this constitutive equation contains terms that are third and fourth order in the principal extensions.

How does this equation differ from the constitutive eqn (4.10) which is second order in the Biot strain? Equation (2.16) gives the Green strain in terms of the Biot strain; incorporation of that expression into (5.22) yields

$$\mathbf{T} = \mathbf{R}\{[\lambda(\mathbf{1}\cdot\mathbf{E}_1)\mathbf{1} + 2\mu\mathbf{E}_1 + (5\mu + n)\mathbf{E}_1^2 + \tfrac{1}{2}(2m - n + \lambda)(1\cdot\mathbf{E}_1^2)\mathbf{1} + (2\lambda - 2\mu + 2m - n)(\mathbf{1}\cdot\mathbf{E}_1)\mathbf{E}_1 + (l + \tfrac{1}{2}n - m - \lambda)(\mathbf{1}\cdot\mathbf{E}_1)^2\mathbf{1}]$$

[4] The constants β_i displayed here are simply related to the constants $\mu\alpha_i$ in Truesdell and Noll (1965). Specifically, $\beta_i = \mu\alpha_i$ for $i = 1, 2, 3$, and $\beta_4 = \mu\alpha_5$, and $\beta_5 = \mu\alpha_6$. Here, the condition that $\alpha_4 + \alpha_5 = 2\alpha_1 - 2\alpha_2$, which is required by the assumption of hyperelasticity, has been incorporated.

$$+[(4\mu+n)\mathbf{E}_1^3+(l+\tfrac{1}{2}n-m-\lambda)(\mathbf{1}\cdot\mathbf{E}_1)(1\cdot\mathbf{E}_1^2)\mathbf{1}$$
$$-(\tfrac{1}{2}n-m)(\mathbf{1}\cdot\mathbf{E}_1^3)\mathbf{1}+(m-\tfrac{1}{2}n-\mu+\lambda)(\mathbf{1}\cdot\mathbf{E}_1)\mathbf{E}_1^2+(m-\tfrac{1}{2}n-\mu+\lambda)(\mathbf{1}\cdot\mathbf{E}_1^2)\mathbf{E}_1]$$
$$+[(\mu+\tfrac{1}{4}n)\mathbf{E}_1^4+\tfrac{1}{4}(l+\tfrac{1}{2}n-m-\lambda)(\mathbf{1}\cdot\mathbf{E}_1^2)^2\mathbf{1}$$
$$-\tfrac{1}{8}(n-2m)(\mathbf{1}\cdot\mathbf{E}_1^4)\mathbf{1}+\tfrac{1}{2}(m-\tfrac{1}{2}n-\mu+\lambda)(\mathbf{1}\cdot\mathbf{E}_1^2)\mathbf{E}_1^2]\}\mathbf{R}^{\mathrm{T}}+o(\mathbf{E}_2^2). \tag{5.23}$$

Note that (5.23) contains (in the first set of square brackets) exactly the second order constitutive eqn (4.10) derived in Section 4.1 for an isotropic material. The coefficients in the two equations are related through $\eta_1=\lambda$, $\eta_2=2\mu$, $\eta_3=3\mu+n$, $\eta_4=\frac{1}{2}(2m-n+\lambda)$, $\eta_5=l+\frac{1}{2}n-m$. In addition, (5.23) includes (in the second set of square brackets) a term that is third order in $\mathbf{E}_1$ and a fourth order term (the third square brackets). Although it includes a number of third and fourth order terms in the extensions, eqn (5.23) clearly is not the general form of a constitutive equation which is fourth order (or even third order) in the extensions. Such an expression would necessarily contain more than five constants. Rather it can be viewed as a constitutive model of a special class of materials which includes all second order terms and only some terms which are third and fourth order in the extensions.

6. Summary

In this paper, the general second order constitutive equation for a hyperelastic material with arbitrary material symmetry is derived in terms of the Biot strain. This strain measure has a clear physical interpretation in that its' eigenvalues are the principal extensions. The general second order equation is specialized to obtain the second order constitutive equations appropriate to the cases of isotropy and transverse isotropy.

The equation obtained here for isotropy is compared with a second order equation obtained by Murnaghan, which is expressed in terms of the Green strain. Although Murnaghan's constitutive equation agrees with the one derived in this paper up to terms that are squared in the principal extensions, it contains additional terms that are third and fourth order in the extensions, but it does not include all such terms.

The use of a strain measure, rather than the displacement gradient, for obtaining a second order constitutive equation has the advantage that the condition of material frame indifference can be easily satisfied, so that the contribution of the rotation can be readily distinguished from that of the strain. However, the most common second order constitutive equation in the literature is that of an isotropic material which is second order in the displacement gradient. For the purpose of comparison, the general constitutive equation for a material of arbitrary symmetry that is second order in the displacement gradient is derived. This general constitutive equation is then specialized to isotropic materials. The resulting equation is shown to be equivalent to the standard second order constitutive equation for isotropic materials given in terms of the displacement gradient.

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References

Adkins, J. E. (1960) Symmetry relations for orthotropic and transversely isotropic materials. *Archive for Rational Mechanics and Analysis* **4,** 193–213.

Dieudonne, J. (1960) *Foundations of Modern Analysis*. Academic Press.

Gurtin, M. (1984) *Introduction to Continuum Mechanics*. Academic Press.

Haughton and Lindsay (1993) The second-order deformations of a finite compressible isotropic elastic annulus subjected to circular shearing. *Proceedings of the Royal Society of London A* **442,** 621–639.

Haughton and Lindsay (1994) The second order deformation of a finite incompressible isotropic elastic annulus subjected to circular shearing. *Acta Mechanica* **104,** 125–141.

Hoger, A. (1993a) Residual stress in an elastic body: a theory for small strains and arbitrary rotations. *Journal of Elasticity* **31,** 1–24.

Hoger, A. (1993b) The elasticity tensors of a residually stressed material. *Journal of Elasticity* **31,** 219–237.

Murnaghan, F. D. (1937) Finite deformations of an elastic solid. *American Journal of Mathematics* **59,** 235–260.

Murnaghan, F. D. (1951) *Finite Deformation of an Elastic Solid*. Dover.

Ogden, R. W. (1984) *Non-Linear Elastic Deformations*. Ellis Horwood.

Pipkin, A. C. and Wineman, A. S. (1962) Material symmetry restrictions on non-polynomial constitutive equations. *Archive for Rational Mechanics and Analysis* **12,** 420–426.

Rivlin, R. S. (1953) The solution of problems in second order elasticity theory. *Journal of Rational Mechanics and Analysis* **2,** 53–81.

Rivlin, R. S. (1955) Further remarks on the stress-deformation relations for isotropic materials. *Journal of Rational Mechanics and Analysis* **4,** 681–701.

Sheng, P.-L. (1955) *Secondary Elasticity*. Chinese Association of Advanced Science, Monograph Series 1, I, No. 1.

Spencer, A. J. M. (1971) Theory of invariants. *Continuum Physics* 1, ed. A. C. Eringen. Academic Press.

Toupin, R. A. and Bernstein, B. (1961) Sound waves in deformed perfectly elastic materials. Acousto-elastic effect. *Journal of the Acoustical Society of America* **33,** 216–225.

Truesdell, C. and Noll, W. (1965) The non-linear field theories of mechanics. *Handbuch der Physik* **III/3**. Springer-Verlag.

Truesdell, C. and Toupin, T. (1960) The classical field theories. *Handbuch der Physik* **III/1**. Springer-Verlag.

Wineman, A. S. and Pipkin, A. C. (1965) Material symmetry restrictions on constitutive equations. *Archive for Rational Mechanics and Analysis* **17,** 184–214.